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Roe Matrices for Ideal MHD and Systematic Construction of Roe Matrices for Systems of Conservation Laws

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In this paper, the construction of a Roe's scheme for the conservative system of ideal magnetohydrodynamics (MHD) is presented. As this method relies on the computation of a Roe matrix, the problem is to find a matrix $\boldsymbol{A}(\boldsymbol{U}_l, \boldsymbol{U}_r)$ which satisfies the following properties. It is required to be consistent with the jacobian of the flux $\boldsymbol{F}$, to have real eigenvalues, a complete set of eigenvectors and to satisfy the relation: $\Delta \boldsymbol{F} = \boldsymbol{A}(\boldsymbol{U}_l, \boldsymbol{U}_r)\, \Delta \boldsymbol{U}$, where $\boldsymbol{U}_l$ and $\boldsymbol{U}_r$ are two admissible states and $\Delta \boldsymbol{U}$ their difference. For the ideal MHD system, using eulerian coordinates, a Roe matrix is obtained without any hypothesis on the specific heat ratio. Especially, its construction relies on an original expression of the magnetic pressure jump. Moreover, a Roe matrix is computed for lagrangian ideal MHD, by extending the results of Munz who obtained such a matrix for the system of lagrangian gas dynamics. So this second matrix involves arithmetic averages unlike the eulerian one, which contains classical Roe averages like in eulerian gas dynamics. In this paper, a systematic construction of lagrangian Roe matrices in terms of eulerian Roe matrices for a general system of conservation laws is also presented. This result, applied to the above eulerian and lagrangian matrices for ideal MHD, gives two new matrices for this system. In the same way, by applying this construction to the gas dynamics equations new Roe matrices are also obtained. All these matrices allow the construction of Roe type schemes. Some numerical examples on the shock tube problem show the applicability of this method.

1. INTRODUCTION

During the last years, Godunov-type schemes became famous for solving hyperbolic systems of conservation laws, with discontinuous solutions. Recently, some of them, originally developed for gas dynamics problems, have been extended to solve ideal magnetohydrodynamics equations.

Especially Daï and Woodward [1, 2] have constructed an approximate Riemann solver for the PPM scheme [5–7]. Zachary and Colella have considered a modification [3, 4] of the Engquist–Osher flux [8], and Khanfir [9] has introduced a formal extension of a kinetic scheme.

Many authors have considered Roe-type schemes to solve ideal MHD. The first important improvement in this subject has been made by Brio and Wu [11]. They have used the Roe scheme and have computed a Roe average, but for the special case where the specific heat ratio γ is equal to 2. Then Powell [10] has given a new formulation of the multidimensional ideal MHD equations, in order to satisfy numerically the relation on the magnetic field divergency. Finally, Aslan has proposed a fluctuation approach, using "partial Roe averaging" [19] and Ryu and Jones describe a Roe-type scheme using arithmetic averages in [35].

In this paper, we are interested in the construction of a Roe scheme for the ideal MHD equations, using eulerian or lagrangian coordinates, without any hypothesis on the value of γ.

Our motivation in using a Roe scheme to solve the MHD equations relies on several advantages which make it well known. First of all, during the last decade, it has known a real success and has been largely used in computational fluid dynamics [28]. Its relative simplicity, its accuracy and its robustness have been greatly demonstrated in order to solve the Euler equations. In the MHD case, its cost is clearly weaker than for schemes similar of those developed in [1–4]. On the other hand, its interest can seem to be less evident when it is compared with an approximate Roe-type scheme involving, for example, arithmetic averages as in [19, 35]. In fact, the extra cost of an exact Roe scheme with respect to this type of scheme is only the evaluation of averaged quantities like density, velocity, magnetic field, and enthalpy with Roe-type averages instead of arithmetic averages. But, this cost is really much weaker than that due to the evaluation of eigenvalues, right eigenvectors, and characteristic variables which must be made in any case. This argument and its intrinsic properties show the interest to use an exact Roe scheme. In particular, the Roe scheme is a Godunov type scheme because it satisfies the "conservation in the small." It follows from this property that Roe scheme captures exactly a stationary discontinuity. Consequently, it is less dissipative than approximate Roe-type solvers. But the capture of this type of discontinuity whether or not it is an admissible discontinuity is the

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counterpart of this property. To handle this difficulty, many authors [28, 34] have proposed some well-known minor correction. Another important feature of Roe scheme is that it is really an upwind scheme in the sense of [33]. The experience in the Euler equations shows that this property is interesting for the treatment of boundary conditions or a fast convergence to stationary solutions in aerodynamic calculations. In MHD, this last point can be attractive to simulate different astrophysical situations like the solar wind flow around comets.

The plan of this paper is as follows. In Section 2, the equations of ideal MHD are defined. In Section 3, the different steps of Roe scheme, which relies on the computation of a Roe matrix, are summarized. Then, the construction of two Roe matrices for ideal MHD, using eulerian (Section 4) and lagrangian (Section 5) coordinates, with some numerical experiments is presented. The treatment of magnetic terms involves an original relation, which represents the main result for this fourth section. For the lagrangian system, Munz' [12] scheme is extended. He has obtained a Roe matrix which introduces arithmetic averages for lagrangian gas dynamics with a general state equation. In Section 6, a systematic construction of lagrangian Roe matrices is given in terms of the eulerian Roe matrices for general hyperbolic conservation laws, in one space dimension. This result is used in Section 7 to construct two new Roe matrices for ideal MHD from the ones computed in Sections 4 and 5. Some numerical examples are also presented. A summary of this study and some remarks are given at the end of this paper in Section 8.

2. IDEAL MHD EQUATIONS

The MHD equations characterize the flow of a conducting fluid in the presence of a magnetic field. They represent the coupling of fluid dynamical equations with Maxwell's equations of electrodynamics. By neglecting displacement current, electrostatic forces, effects of viscosity, resistivity and heat conduction, one obtains the ideal MHD equations [13]:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla\cdot(\rho \boldsymbol{V}) = 0 \\[2ex] \dfrac{\partial}{\partial t}(\rho \boldsymbol{V}) + \nabla\cdot\left(\rho \boldsymbol{V}\boldsymbol{V} + p^*\boldsymbol{I} - \dfrac{1}{\mu}\boldsymbol{B}\boldsymbol{B}\right) = \boldsymbol{0} \\[2ex] \dfrac{\partial \boldsymbol{B}}{\partial t} + \nabla\cdot(\boldsymbol{V}\boldsymbol{B} - \boldsymbol{B}\boldsymbol{V}) = \boldsymbol{0} \\[2ex] \dfrac{\partial}{\partial t}(\rho E^*) + \nabla\cdot\left(\rho H^* \boldsymbol{V} - \dfrac{1}{\mu}(\boldsymbol{B}\cdot\boldsymbol{V})\boldsymbol{B}\right) = 0 \end{cases} \tag{2.1}$$

with the constraint $\nabla\cdot\boldsymbol{B} = 0$, which is satisfied by the solution of the initial value problem if it is satisfied initially.

Here, ρ is the density, $\boldsymbol{V}$ is the velocity, $\boldsymbol{B}$ is the magnetic field, p^* is the full pressure, ρE^* is the energy, ρH^* is the enthalpy, and μ is the vacuum permittivity.

The pressure, energy, and enthalpy are defined by

$$\begin{aligned} p^* &= p + \frac{\boldsymbol{B}^2}{2\mu}, \quad \rho E^* = \frac{1}{2}\rho \boldsymbol{V}^2 + \rho\varepsilon + \frac{\boldsymbol{B}^2}{2\mu}, \\ \rho\varepsilon &= \frac{p}{\gamma - 1}, \qquad \rho H^* = \rho E^* + p^*, \end{aligned} \tag{2.2}$$

where p is the hydrodynamic pressure and $\rho\varepsilon$ is the internal energy. In the following, we assume $\mu = 1$.

3. A ROE SCHEME REVIEW

Let us consider the following hyperbolic system of conservation laws:

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial}{\partial x}\boldsymbol{F}(\boldsymbol{U}) = \boldsymbol{0}. \tag{3.1}$$

The numerical solution $\boldsymbol{U}^n$ at time t^n is assumed to be a piecewise constant function on each grid cell $x_{i-1/2} < x \le x_{i+1/2}$. The resolution of (3.1) with the initial condition $\boldsymbol{U}^n$ on the time interval $t^n \le t \le t^{n+1}$ defines a sequence of Riemann problems at each interface of the grid.

In a finite volume discretization, the solution $\boldsymbol{U}_i^{n+1}$ is obtained by averaging the exact or approximated solution $\hat{\boldsymbol{U}}$ of (3.1) at the discrete time level t^{n+l},

$$\boldsymbol{U}_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \hat{\boldsymbol{U}}(x, t^{n+1})\, dx, \tag{3.2}$$

where Δx denotes the uniform size of the computational cells. These values are then used to define new piecewise constant function at the time level t^{n+1}.

Instead of computing the exact solution of the system (3.1) like in Godunov's scheme [14], the Roe's method [15] consists in solving exactly the following linearized Riemann problem at each cell interface:

$$\begin{aligned} &\frac{\partial \boldsymbol{U}}{\partial t} + \overline{\boldsymbol{A}}_{i+1/2}\frac{\partial \boldsymbol{U}}{\partial x} = \boldsymbol{0} \\ &\boldsymbol{U}(x, t_n) = \begin{cases} \boldsymbol{U}_i^n, & \text{if } x < x_{i+1/2}, \\ \boldsymbol{U}_{i+1}^n, & \text{if } x \ge x_{i+1/2}. \end{cases} \end{aligned} \tag{3.3}$$

The matrix $\overline{\boldsymbol{A}}_{i+1/2}$ is called a Roe matrix and is required to have the properties:

$$\boldsymbol{F}(\boldsymbol{U}_{i+1}) - \boldsymbol{F}(\boldsymbol{U}_i) = \overline{\boldsymbol{A}}_{i+1/2}(\boldsymbol{U}_{i+1} - \boldsymbol{U}_i) \tag{3.4}$$

as

$$\boldsymbol{U}_i, \boldsymbol{U}_{i+1} \rightarrow \boldsymbol{U}_0, \quad \overline{\boldsymbol{A}}_{i+1/2}(\boldsymbol{U}_i, \boldsymbol{U}_{i+1}) \rightarrow \boldsymbol{A}(\boldsymbol{U}_0), \tag{3.5}$$

where $\boldsymbol{A} = d\boldsymbol{F}/d\boldsymbol{U}$

$$\overline{\boldsymbol{A}}_{i+1/2} \text{ has real eigenvalues and a complete set of eigenvectors.} \tag{3.6}$$

The solution of (3.3) is averaged over each cell of the mesh in order to obtain the numerical approximation at next time level t^{n+1},

$$\boldsymbol{U}_i^{n+1} = \boldsymbol{U}_i^n - \frac{\Delta t}{\Delta x}(\boldsymbol{H}_{i+1/2}^n - \boldsymbol{H}_{i-1/2}^n), \tag{3.7}$$

where Δt is the time step.

The conservative form (3.7) involves the numerical flux at cell interfaces, which is given by the relation [15]

$$\boldsymbol{H}_{i+1/2}^n = \frac{1}{2}\left(\boldsymbol{F}_i + \boldsymbol{F}_{i+1} - \sum_k |\lambda_{i+1/2}^k| \eta_{i+1/2}^k \boldsymbol{R}_{i+1/2}^k\right), \tag{3.8}$$

where $\lambda_{i+1/2}^k$ denote the eigenvalues of $\overline{\boldsymbol{A}}_{i+1/2}$ and its characteristic variables are defined by $\eta_{i+1/2}^k = \boldsymbol{L}_{i+1/2}^k \cdot \Delta \boldsymbol{U}$. Moreover, $\boldsymbol{R}_{i+1/2}^k$ and $\boldsymbol{L}_{i+1/2}^k$ are respectively its right and left eigenvectors.

So the main step in Roe's scheme construction is the computation of a Roe matrix. Now, this is what we develop for the ideal MHD system in one space dimension.

4. A ROE MATRIX FOR EULERIAN IDEAL MHD

4.1. *The Model and Some Properties*

In one space dimension, the ideal MHD system, using eulerian coordinates, is given by the equations [13, 17, 26]

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\rho u = 0\\
&\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + p^*) = 0\\
&\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho u v - B_x B_y) = 0\\
&\frac{\partial}{\partial t}(\rho w) + \frac{\partial}{\partial x}(\rho u w - B_x B_z) = 0\\
&\frac{\partial B_y}{\partial t} + \frac{\partial}{\partial x}(B_y u - B_x v) = 0\\
&\frac{\partial B_z}{\partial t} + \frac{\partial}{\partial x}(B_z u - B_x w) = 0\\
&\frac{\partial}{\partial t}(\rho E^*) + \frac{\partial}{\partial x}(\rho u H^* - B_x(B_x u + B_y v + B_z w)) = 0,
\end{aligned} \tag{4.1}$$

where u, v, w are the three components of the velocity $\boldsymbol{V}$ and B_x, B_y, B_z are the components of the magnetic field $\boldsymbol{B}$.

The magnetic field has to satisfy $\nabla \cdot \boldsymbol{B} = 0$, which in one dimension becomes $B_x = \text{const}$. This system is hyperbolic and has seven eigenvalues, written in an increasing order,

$$u - c_f, u - c_a, u - c_s, u, u + c_s, u + c_a, u + c_f, \tag{4.2}$$

with

$$\begin{aligned}
c_a^2 &= b_x^2\\
c_f^2 &= \frac{1}{2}((a^*)^2 + \sqrt{(a^*)^4 - 4a^2 b_x^2})\\
c_s^2 &= \frac{1}{2}((a^*)^2 - \sqrt{(a^*)^4 - 4a^2 b_x^2})
\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}
\boldsymbol{b} &= \frac{\boldsymbol{B}}{\sqrt{\rho}}, \quad \boldsymbol{b} = {}^t(b_x, b_y, b_z)\\
(a^*)^2 &= a^2 + \boldsymbol{b}^2, \quad a^2 = \gamma\frac{p}{\rho}.
\end{aligned} \tag{4.4}$$

The quantities c_a, c_s, c_f define the velocities of the Alfvén waves, the slow and fast waves.

It is interesting to note that this system is not strictly hyperbolic; some eigenvalues may coincide especially when $B_x = 0$ or $B_y^2 + B_z^2 = 0$. Brio and Wu [11] have studied a renormalization of the eigenvectors given by Jeffrey and Taniuti [13] to obtain a complete set of eigenvectors even in the two above cases. But the components of these eigenvectors remain singular in the magnetosonic case, arising when the fast, slow, and Alfvén speeds coincide. Roe and Balsara [20] have examined the eigenstructure of the ideal MHD equations. Especially, their study shows that the indeterminacy near the magnetosonic case does not induce any difficulty to compute a numerical flux even if it cannot be avoided.

We conclude this section with an important property of the system when $\gamma = 2$. In this case, the system decouples into two systems. The first one looks like the gas dynamic system, with p^* as pressure, ρE^* as energy, ρH^* as enthalpy, and the following state equation:

$$p^* = \rho\varepsilon + \frac{\boldsymbol{B}^2}{2}.$$

The second system, which is a set of advection equations, describes the evolution of the transverse components of the magnetic field.

Brio and Wu [11] have used this property to develop a Roe-type scheme for ideal MHD. Let us denote $\boldsymbol{U}_l$ and $\boldsymbol{U}_r$ two admissible states. For the special case $\gamma = 2$, they have computed a Roe matrix $\overline{\boldsymbol{A}}$ which is defined by the relation

$$\overline{\boldsymbol{A}}(\boldsymbol{U}_l, \boldsymbol{U}_r) = \boldsymbol{A}(\boldsymbol{U}_{ave}(\boldsymbol{U}_l, \boldsymbol{U}_r))$$

where $\boldsymbol{U}_{ave}(\boldsymbol{U}_l, \boldsymbol{U}_r)$ is a generalization of the Roe average for gas dynamics,

$$\boldsymbol{U}_{ave}(\boldsymbol{U}_l, \boldsymbol{U}_r) = \boldsymbol{U}(\underline{\rho}, \overline{u}, \overline{v}, \overline{w}, \overline{H^*}, \underline{B}_y, \underline{B}_z), \tag{4.5}$$

with

$$\overline{\xi} = \frac{\sqrt{\rho_l}\xi_l + \sqrt{\rho_r}\xi_r}{\sqrt{\rho_l} + \sqrt{\rho_r}}, \quad \underline{\xi} = \frac{\sqrt{\rho_r}\xi_l + \sqrt{\rho_l}\xi_r}{\sqrt{\rho_l} + \sqrt{\rho_r}}. \tag{4.6}$$

For the general case, they have not found a Roe matrix; they have used the jacobian matrix at some average state but the property (3.4) is not satisfied.

Now, in order to understand the difficulties and the results we obtain, the computation of a Roe matrix on a simplified model of ideal MHD is presented.

4.2. *A Roe Matrix for Ideal Isentropic MHD*

The model is given by the equations

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\rho u = 0\\
&\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}\left(\rho u^2 + p + \frac{B^2}{2}\right) = 0\\
&\frac{\partial B}{\partial t} + \frac{\partial}{\partial x}(Bu) = 0,
\end{aligned} \tag{4.7}$$

where the pressure is given by the isentropic law:

$$p = C\rho^\gamma \quad \text{with } C > 0; \gamma \geq 1.$$

Here, the quantity B represents one of the transverse components of the magnetic field. The other ones are assumed to be zero, like the transverse components of the velocity.

This system looks like the isentropic gas dynamics model, with an advection equation for the magnetic field and the presence of the magnetic pressure in the momentum conservation equation.

In order to construct a Roe matrix for this system, we first use the analogy with gas dynamics. When the magnetic field is zero, the matrix is required to coincide with the Roe matrix for gas dynamics. This constraint gives us the averages to use in order to express the jump on the hydrodynamic terms.

For the following, we define $\Delta\xi = \xi_l - \xi_r$ and we use the classical relation

$$\Delta(\xi\eta) = \underline{\xi}\,\Delta\eta + \overline{\eta}\,\Delta\xi, \tag{4.8}$$

where $\underline{\xi}$ and $\overline{\eta}$ are defined by (4.6).

We introduce the positive average quantity $\overline{a}^2$ for the sound velocity. It is a function of $\boldsymbol{U}_l$ and $\boldsymbol{U}_r$ and it is defined such that $\Delta p = \overline{a}^2\,\Delta\rho$.

The most important difficulty lies in the decomposition of the magnetic pressure jump.

The natural relation $\Delta B^2 = 2\tilde{B}\,\Delta B$, where $\tilde{B}$ denotes the arithmetic average between the left and right states, leads to a matrix whose eigenvalues are not real. So, we are looking for a more general relation which expresses the variations of the magnetic pressure in terms of each component of the conservative vector,

$$\Delta\frac{B^2}{2} = X\,\Delta\rho + Y\,\Delta(\rho u) + Z\,\Delta B, \tag{4.9}$$

where X, Y, Z are three coefficients to evaluate.

These quantities are computed such that the resulting matrix $\overline{\boldsymbol{A}}^{\text{isen}}$ is a Roe matrix:

$$\overline{\boldsymbol{A}}^{\text{isen}} = \begin{bmatrix} 0 & 1 & 0\\ -\overline{u}^2 + \overline{a}^2 + X & 2\overline{u} + Y & Z\\ -\overline{u}\,\overline{\left(\dfrac{B}{\rho}\right)} & \overline{\left(\dfrac{B}{\rho}\right)} & \overline{u} \end{bmatrix}. \tag{4.10}$$

Like the jacobian of the continuous system (4.7), it is natural to look for a matrix whose eigenvalues respect the galilean invariance and are symmetrical with respect to the velocity $\overline{u}$. This requirement implies that the unknown parameter Y is zero.

Moreover, we want to agree with the results of Brio and Wu in the special case $\gamma = 2$. That is why we use the average $\underline{B}$ for the magnetic field; this is the unique average of B, involved with the Roe matrix of Brio and Wu. This leads to the relation

$$\Delta\frac{B^2}{2} = X\,\Delta\rho + \underline{B}\,\Delta B. \tag{4.11}$$

Hence, we easily obtain

$$X = \frac{(\Delta B)^2}{2(\sqrt{\rho_l} + \sqrt{\rho_r})^2}. \tag{4.12}$$

The matrix defined by the relations (4.9) to (4.12) is a Roe matrix, whose eigenvalues are real and given by the expressions

$$\overline{u} - \sqrt{\overline{a}^2 + \underline{B}^2/\underline{\rho} + X}, \overline{u}, \overline{u} + \sqrt{\overline{a}^2 + \underline{B}^2/\underline{\rho} + X}. \tag{4.13}$$

We remark that this matrix coincides with the one of isentropic gas dynamics when the magnetic field is zero. This result is very interesting because it gives a way to express the jump on the magnetic pressure with the relation (4.11).

Let us note that this relation is quite surprising: it expresses the magnetic pressure jump not only in terms of the magnetic field jump, but also in terms of the density jump, which is less natural.

4.3. *A Roe Matrix for Ideal MHD*

We present here the construction of a Roe matrix for the system (4.1). It is computed without any hypothesis on γ. In order to compute this matrix, we use the main results obtained in the above section.

The jumps of the hydrodynamic terms in the flux are treated with the same relations and the same averages as the gas dynamics equations. For example, for the variations of the kinetic energy, we use

$$\Delta(\rho \boldsymbol{V}^2) = -\overline{\boldsymbol{V}}^2 \Delta\rho + 2\overline{\boldsymbol{V}} \cdot \Delta(\rho \boldsymbol{V}). \tag{4.14}$$

On the other hand, the variations of the magnetic pressure are expressed in terms of the magnetic field jump and the density jump [21],

$$\Delta \frac{\boldsymbol{B}^2}{2} = X \Delta\rho + \underline{\boldsymbol{B}} \cdot \Delta \boldsymbol{B}. \tag{4.15}$$

With the relations (4.15), (4.12), and (4.14), we can easily express the jumps on the pressure and the full enthalpy:

$$\Delta p = (\gamma - 1)\left[\left(\frac{\overline{\boldsymbol{V}}^2}{2} - X\right)\Delta\rho - \overline{\boldsymbol{V}} \cdot \Delta(\rho \boldsymbol{V}) + \Delta(\rho E^*) - \underline{\boldsymbol{B}} \cdot \Delta \boldsymbol{B}\right]$$

$$\Delta(\rho H^*) = \left[(2-\gamma)X + \frac{1}{2}(\gamma - 1)\overline{\boldsymbol{V}}^2\right]\Delta\rho - (\gamma - 1)\overline{\boldsymbol{V}} \cdot \Delta \boldsymbol{V} + \gamma\, \Delta(\rho E^*) + (2-\gamma)\underline{\boldsymbol{B}} \cdot \Delta \boldsymbol{B}.$$

By this way, we obtain the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \delta_{21} & \delta_{22} & \delta_{23} & \delta_{24} & \delta_{25} & \delta_{26} & \delta_{27} \\ -\overline{u}\,\overline{v} & \overline{v} & \overline{u} & 0 & -B_x & 0 & 0 \\ -\overline{u}\,\overline{w} & \overline{w} & 0 & \overline{u} & 0 & -B_x & 0 \\ \frac{-\underline{B_y}}{\underline{\rho}}\overline{u} + \frac{B_x}{\underline{\rho}}\overline{v} & \frac{\underline{B_y}}{\underline{\rho}} & \frac{-B_x}{\underline{\rho}} & 0 & \overline{u} & 0 & 0 \\ \frac{-\underline{B_z}}{\underline{\rho}}\overline{u} + \frac{B_x}{\underline{\rho}}\overline{w} & \frac{\underline{B_z}}{\underline{\rho}} & 0 & \frac{-B_x}{\underline{\rho}} & 0 & \overline{u} & 0 \\ \delta_{71} & \delta_{72} & \delta_{73} & \delta_{74} & \delta_{75} & \delta_{76} & \delta_{77} \end{bmatrix} \tag{4.16}$$

with

$$\delta_{21} = -\overline{u}^2 + (2-\gamma)X + \frac{\gamma - 1}{2}\overline{\boldsymbol{V}}^2, \delta_{22} = 2\overline{u} - (\gamma - 1)\overline{u}$$

$$\delta_{23} = -(\gamma - 1)\overline{v}, \delta_{24} = -(\gamma - 1)\overline{w}$$

$$\delta_{25} = (2-\gamma)\underline{B_y}, \delta_{26} = (2-\gamma)\underline{B_z}, \delta_{27} = \gamma - 1$$

and:

$$\delta_{71} = -\overline{u}\overline{H^*} + \overline{u}(\delta_{21} + \overline{u}^2) + \frac{B_x}{\underline{\rho}}(\overline{\boldsymbol{V}} \cdot \underline{\boldsymbol{B}})$$

$$\delta_{72} = \overline{H^*} + \overline{u}(\delta_{22} - 2\overline{u}) - \frac{B_x^2}{\underline{\rho}}, \delta_{73} = \overline{u}\delta_{23} - \frac{B_x}{\underline{\rho}}\underline{B_y},$$

$$\delta_{74} = \overline{u}\delta_{24} - \frac{B_x}{\underline{\rho}}\underline{B_z}$$

$$\delta_{75} = \overline{u}\delta_{25} - B_x\overline{v}, \delta_{76} = \overline{u}\delta_{26} - B_x\overline{w}, \delta_{77} = \overline{u} + \overline{u}\delta_{27}$$

The above matrix is a Roe matrix and its eigenvalues are given in an increasing order,

$$\overline{u} - \overline{c}_f, \overline{u} - \overline{c}_a, \overline{u} - \overline{c}_s, \overline{u}, \overline{u} + \overline{c}_s, \overline{u} + \overline{c}_a, \overline{u} + \overline{c}_f$$

with

$$\bar{c}_a^2 = \overline{b_x}^2$$
$$\bar{c}_f^2 = \frac{1}{2}\left((\overline{a^*})^2 + \sqrt{(\overline{a^*})^4 - 4\bar{a}^2\overline{b_x}^2}\right) \qquad (4.17)$$
$$\bar{c}_s^2 = \frac{1}{2}\left((\overline{a^*})^2 - \sqrt{(\overline{a^*})^4 - 4\bar{a}^2\overline{b_x}^2}\right)$$

and

$$\bar{\boldsymbol{b}} = \frac{\underline{\boldsymbol{B}}}{\sqrt{\underline{\rho}}}, \quad \bar{\boldsymbol{b}} = {}^t(\overline{b_x}, \overline{b_y}, \overline{b_z})$$

$$\overline{a^*}^2 = \bar{a}^2 + \bar{\boldsymbol{b}}^2, \quad \bar{a}^2 = (2-\gamma)X + (\gamma - 1)\left(\overline{H^*} - \frac{\overline{\boldsymbol{V}}^2}{2} - \bar{\boldsymbol{b}}^2\right).$$

In order to define completely, the Roe numerical flux, the eigenvectors, and the characteristic variables are required. The eigenvectors are defined by

$$\boldsymbol{R}_{\bar{u}} = \frac{1}{\bar{a}^2}\begin{bmatrix} 1 \\ \bar{u} \\ \bar{v} \\ \bar{w} \\ 0 \\ 0 \\ \dfrac{\overline{\boldsymbol{V}}^2}{2} + \left[\dfrac{\gamma-2}{\gamma-1}\right]X \end{bmatrix} \qquad (4.18)$$

$$\boldsymbol{R}_{\bar{u}\pm\bar{c}_a} = \begin{bmatrix} 0 \\ 0 \\ \pm\underline{\rho}\beta_z \\ \mp\underline{\rho}\beta_y \\ -S\sqrt{\underline{\rho}}\beta_z \\ S\sqrt{\underline{\rho}}\beta_y \\ \pm\underline{\rho}(\bar{v}\beta_z - \bar{w}\beta_y) \end{bmatrix}$$

$$\boldsymbol{R}_{\bar{u}\pm\bar{c}_s} = \frac{1}{\underline{\rho}\bar{a}^2}\begin{bmatrix} \underline{\rho}\alpha_s \\ \underline{\rho}\alpha_s(\bar{u} \pm \bar{c}_s) \\ \underline{\rho}(\alpha_s\bar{v} \pm \alpha_f\bar{c}_f\beta_y S) \\ \underline{\rho}(\alpha_s\bar{w} \pm \alpha_f\bar{c}_f\beta_z S) \\ -\sqrt{\underline{\rho}}\alpha_f\bar{a}\beta_y \\ -\sqrt{\underline{\rho}}\alpha_f\bar{a}\beta_z \\ \underline{\rho}\alpha_s\left[\overline{H}^* - \dfrac{\underline{\boldsymbol{B}}^2}{\underline{\rho}} \pm \bar{u}\,\bar{c}_s\right] \pm \underline{\rho}\alpha_f\bar{c}_f S(\bar{v}\beta_y + \bar{w}\beta_z) - \sqrt{\underline{\rho}}\alpha_f\bar{a}|\underline{\boldsymbol{B}}_\perp| \end{bmatrix} \qquad (4.19)$$

$$\boldsymbol{R}_{\bar{u}\pm\bar{c}_f} = \frac{1}{\underline{\rho}\bar{a}^2}\begin{bmatrix} \underline{\rho}\alpha_f \\ \underline{\rho}\alpha_f(\bar{u} \pm \bar{c}_f) \\ \underline{\rho}(\alpha_f\bar{v} \mp \alpha_s\bar{c}_s\beta_y S) \\ \underline{\rho}(\alpha_f\bar{w} \mp \alpha_s\bar{c}_s\beta_z S) \\ \sqrt{\underline{\rho}}\alpha_s\bar{a}\beta_y \\ \sqrt{\underline{\rho}}\alpha_s\bar{a}\beta_z \\ \underline{\rho}\alpha_f\left[\overline{H}^* - \dfrac{\underline{\boldsymbol{B}}^2}{\underline{\rho}} \pm \bar{u}\,\bar{c}_f\right] \mp \underline{\rho}\alpha_s\bar{c}_s S(\bar{v}\beta_y + \bar{w}\beta_z) - \sqrt{\underline{\rho}}\alpha_s\bar{a}|\underline{\boldsymbol{B}}_\perp| \end{bmatrix}$$

and the characteristic variables satisfy the relations

$$
\begin{aligned}
\eta_{\bar{u}} &= [\bar{a}^2 - X]\,\Delta\rho - \Delta p \\
\eta_{\bar{u}\pm\bar{c}_a} &= \frac{1}{2}\left[\mp\beta_y\,\Delta w \pm \beta_z\,\Delta v + \frac{S}{\sqrt{\underline{\rho}}}(\beta_y\,\Delta B_z - \beta_z\,\Delta B_y)\right] \\
\eta_{\bar{u}\pm\bar{c}_s} &= \frac{1}{2}[\alpha_s[X\,\Delta\rho + \Delta p] \pm \underline{\rho}\alpha_f\bar{c}_f S(\beta_y\,\Delta v + \beta_z\,\Delta w) \\
&\quad \pm \underline{\rho}\alpha_s\bar{c}_s\,\Delta u - \sqrt{\underline{\rho}}\alpha_f\bar{a}(\beta_y\,\Delta B_y + \beta_z\,\Delta B_z)] \\
\eta_{\bar{u}\pm\bar{c}_f} &= \frac{1}{2}[\alpha_f[X\,\Delta\rho + \Delta p] \mp \underline{\rho}\alpha_s\bar{c}_s S(\beta_y\,\Delta v + \beta_z\,\Delta w) \\
&\quad \pm \underline{\rho}\alpha_f\bar{c}_f\Delta u + \sqrt{\underline{\rho}}\alpha_s\bar{a}(\beta_y\,\Delta B_y + \beta_z\,\Delta B_z)]
\end{aligned}
\tag{4.20}
$$

with the notations

$$
\begin{aligned}
&S = \operatorname{sign}(B_x), \quad \beta_{y,z} = \frac{\underline{B}_{y,z}}{|\underline{\boldsymbol{B}}_\perp|}, \quad |\underline{\boldsymbol{B}}_\perp| = \sqrt{\underline{B}_y^2 + \underline{B}_z^2} \\
&\alpha_f^2 = \frac{\bar{a}^2 - \bar{c}_s^2}{\bar{c}_f^2 - \bar{c}_s^2}, \quad \alpha_s^2 = \frac{\bar{c}_f^2 - \bar{a}^2}{\bar{c}_f^2 - \bar{c}_s^2}
\end{aligned}
\tag{4.21}
$$

Let us note that the above eigenvectors have been obtained by the same normalization as in Roe–Balsara [20]. Moreover, the treatment of the indeterminate cases is the same as in [20]. So, we have a complete set of eigenvectors in all cases.

4.4. *Numerical Application*

With this matrix, we construct a first-order Roe type scheme. For the numerical example, we choose a coplanar MHD Riemann problem, whose initial value consists of two constant states $\boldsymbol{U}_l$ and $\boldsymbol{U}_r$. The initial left and right values have been suggested by Brio and Wu in [11] and are commonly used to test numerical schemes for one-dimensional ideal MHD:

$$
\begin{aligned}
(\rho, u, v, w, B_y, B_z, p)_l &= (1, 0, 0, 0, 1, 0, 1), \\
(\rho, u, v, w, B_y, B_z, p)_r &= (\tfrac{1}{8}, 0, 0, 0, -1, 0, \tfrac{1}{10}).
\end{aligned}
\tag{4.22}
$$

Moreover, $B_x = \frac{3}{4}$ and $\gamma = 1.4$.

Note that the hydrodynamical data used here are identical to those in Sod's shock tube Riemann problem. The discretization is given by 800 cells whose length is 1. The initial discontinuity is located at the middle of the mesh.

In [11], Brio and Wu have chosen a value of γ equal to 2. In this special case they have been able to find a Roe matrix which is the flux jacobian evaluated at the averaged state (4.5). For this value of γ, our Roe matrix coincides with Brio and Wu matrix and thus gives exactly the same numerical results.

Here, we choose the value $\gamma = 1.4$. In this case, Brio and Wu have proposed to use a simple averaging procedure but the resulting matrix is not a Roe matrix.

The results are presented on Figs. 1 and 2 at 80 s with a CFL number equal to 0.9. Figure 1 shows the numerical solution computed with the first-order Roe type scheme and Fig. 2 with the Lax–Friedrichs scheme [18]. Here, we can see the efficiency of the Roe scheme which is more accurate than the Lax scheme. As in [11], calculations with an increased number of grid points up to 20,000 show the convergence of the Lax–Friedrichs scheme to the same solution as the one obtained by our Roe scheme.

For each quantity, the solution contains five constant states separated by a fast rarefaction wave, a slow compound wave, a slow shock, and a fast rarefaction wave. The density presents a sixth constant state because this variable is discontinuous across the contact discontinuity.

This numerical example is interesting because it involves a compound wave. This allows us to show off one of the typical features of solutions of the MHD system. Here, the slow compound wave contains a slow shock attached to a slow rarefaction moving to the left. It is due to the change of sign of the component B_y in the initial discontinuity. The detailed study of this wave type has been made by Brio and Wu in [11] where they established a relation between the existence of the compound wave and the nonconvex character of the MHD system.

5. A ROE SCHEME FOR LAGRANGIAN IDEAL MHD

In this section, the construction of a Roe matrix for the equations of ideal MHD, using lagrangian coordinates, is studied. The derivation of the lagrangian formulation can be found in [16].

5.1. *The Model*

In one dimension, the equations of lagrangian ideal MHD may be obtained from the eulerian ones by introducing the new variables (τ, m) such that:

$$
\tau = t \quad \text{and} \quad m = \int \rho\,dx. \tag{5.1}
$$

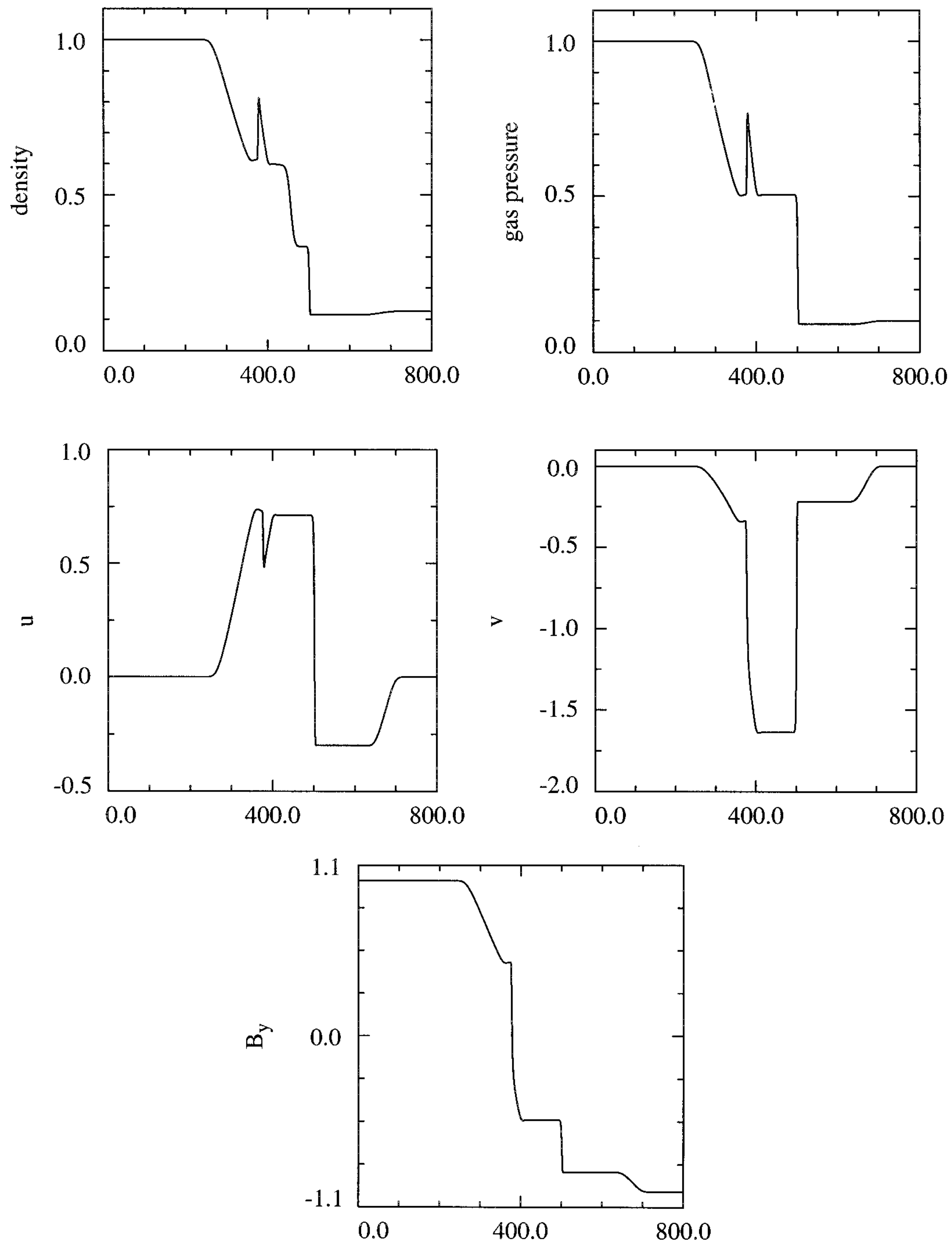

FIG. 1. First-order Roe-type scheme for eulerian coordinates.

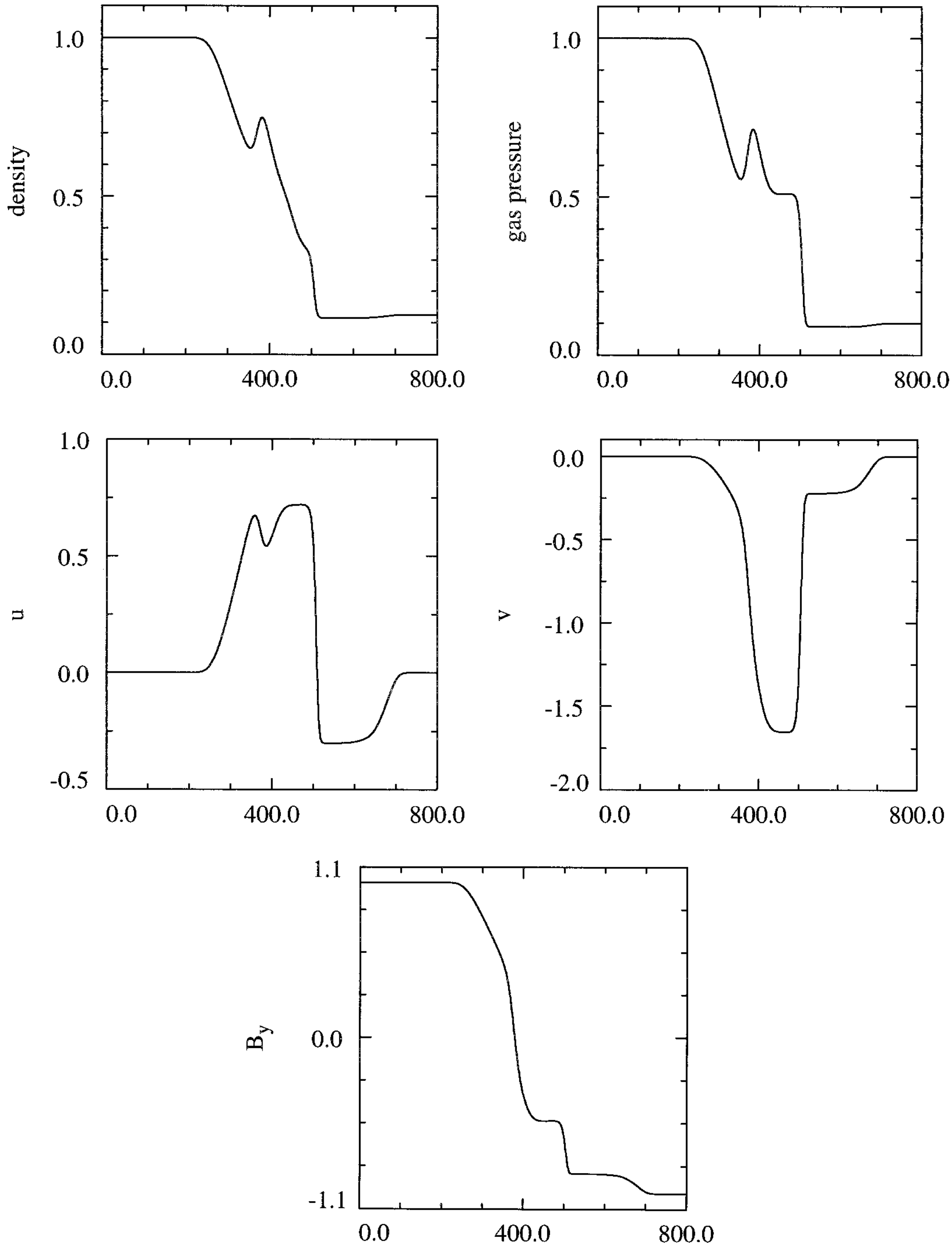

FIG. 2. Lax scheme for eulerian coordinates.

So the system is given by the equations

$$
\begin{aligned}
&\frac{\partial \vartheta}{\partial \tau} - \frac{\partial u}{\partial m} = 0 \\
&\frac{\partial u}{\partial \tau} + \frac{\partial}{\partial m} p^* = 0 \\
&\frac{\partial v}{\partial \tau} + \frac{\partial}{\partial m} (-B_x B_y) = 0 \\
&\frac{\partial w}{\partial \tau} + \frac{\partial}{\partial m} (-B_x B_z) = 0 \\
&\frac{\partial}{\partial \tau} (\vartheta B_y) + \frac{\partial}{\partial m} (-B_x v) = 0 \\
&\frac{\partial}{\partial \tau} (\vartheta B_z) + \frac{\partial}{\partial m} (-B_x w) = 0 \\
&\frac{\partial E^*}{\partial \tau} + \frac{\partial}{\partial m} (u p^* - B_x(B_x u + B_y v + B_z w)) = 0,
\end{aligned}
\tag{5.2}
$$

where ϑ is the specific volume: $\vartheta = 1/\rho$. As in the eulerian case, B_x is a constant.

This system of conservation laws is noted:

$$\frac{\partial \boldsymbol{W}}{\partial \tau} + \frac{\partial \boldsymbol{G}}{\partial m} = \boldsymbol{0}. \tag{5.3}$$

It is hyperbolic and its eigenvalues are given by

$$-\frac{c_f}{\vartheta}, -\frac{c_a}{\vartheta}, -\frac{c_s}{\vartheta}, 0, \frac{c_s}{\vartheta}, \frac{c_a}{\vartheta}, \frac{c_f}{\vartheta} \tag{5.4}$$

with the previous notations (4.3) and (4.4).

5.2. *Construction of the Roe Matrix*

Like the eulerian ideal MHD system, when $\gamma = 2$ and B_x is uniformly zero, (5.2) decouples into two systems. One of them looks like the lagrangian gas dynamics system with p^* as pressure, E^* as energy, and the following equation of state:

$$p^* = \rho\varepsilon + \frac{\boldsymbol{B}^2}{2}.$$

A Roe matrix has been computed by Munz [12] for the lagrangian gas dynamics equations. We recall the main points of Munz results.

In the case of a perfect gas equation of state, the Roe matrix computed by Munz is given by the jacobian of the system evaluated at the averaged state [12] $\boldsymbol{W}_{ave}(\boldsymbol{W}_l^D, \boldsymbol{W}_r^D) = \boldsymbol{W}^D(\tilde{\vartheta}\ \tilde{u}, \tilde{p})$, where $\boldsymbol{W}^D = {}^t(\vartheta, u, E^h)$ and the hydrodynamic energy is given by

$$E^h = \frac{p\vartheta}{\gamma - 1} + \frac{u^2}{2},$$

where $\tilde{\xi}$ denotes the arithmetic average between the left and right states.

Note that this matrix has been already discovered in [27] to obtain “conservation in the small.” This construction relies on an essential relation [12] which expresses the jump on the pressure in terms of each component of the vector $\boldsymbol{W}^D$:

$$\Delta p = -\frac{\tilde{p}}{\tilde{\vartheta}} \Delta\vartheta - (\gamma - 1)\frac{\tilde{u}}{\tilde{\vartheta}} \Delta u + \frac{(\gamma - 1)}{\tilde{\vartheta}} \Delta E^h. \tag{5.5}$$

According to the decoupling property, we are able to find an averaged state for the lagrangian MHD system when $\gamma = 2$ from the Munz averaged state. For this special case, a Roe matrix is given by the jacobian of the lagrangian flux calculated at this state. As this matrix involves arithmetic averages, it seems natural to use the same kind of averages to derive a Roe matrix for any γ.

For the lagrangian ideal MHD model (5.3), let us define $\overline{\boldsymbol{A}}_L$ the Roe matrix we are looking for. The first step of the Roe's method consists in developing the jump on the flux in terms of the variations of the conservative variables:

$$\Delta \boldsymbol{G} = \overline{\boldsymbol{A}}_L(\boldsymbol{W}_l, \boldsymbol{W}_r)\, \Delta \boldsymbol{W}. \tag{5.6}$$

In the following, we often use the classical relation:

$$\Delta(\xi\eta) = \tilde{\xi}\,\Delta\eta + \tilde{\eta}\,\Delta\xi \tag{5.7}$$

in order to express the jump on a product of variables.

Note that the lagrangian coordinates introduce $\vartheta\boldsymbol{B}$ as a conservative variable instead of $\boldsymbol{B}$ with eulerian coordinates. To express the jump on the magnetic field involved by the flux, we apply (5.7) to compute the variations on $\vartheta\boldsymbol{B}$ and we finally get

$$\Delta \boldsymbol{B} = -\frac{\tilde{\boldsymbol{B}}}{\tilde{\vartheta}} \Delta\vartheta + \frac{1}{\tilde{\vartheta}} \Delta(\vartheta \boldsymbol{B}). \tag{5.8}$$

This gives the relation we use to express the jump on the magnetic pressure:

$$\Delta \frac{\boldsymbol{B}^2}{2} = \tilde{\boldsymbol{B}} \cdot \Delta \boldsymbol{B} = -\frac{(\tilde{\boldsymbol{B}})^2}{\tilde{\vartheta}} \Delta\vartheta + \frac{\tilde{\boldsymbol{B}}}{\tilde{\vartheta}} \cdot \Delta(\vartheta \boldsymbol{B}). \tag{5.9}$$

The most difficult term to treat in the flux is the full pressure. By definition, this quantity is the sum of the hydrodynamic pressure p and the magnetic pressure. So we have

$$\Delta p^* = \Delta p + \Delta \frac{\boldsymbol{B}^2}{2}.$$

A generalization of the relation (5.5) obtained by Munz gives

$$\Delta p = -\frac{\tilde{p}}{\tilde{\vartheta}} \Delta \vartheta - (\gamma - 1) \frac{\tilde{\boldsymbol{V}}}{\tilde{\vartheta}} \cdot \Delta \boldsymbol{V} + \frac{(\gamma-1)}{\tilde{\vartheta}} \Delta E^h.$$

Now, the hydrodynamic energy is defined by

$$E^h = \frac{p\vartheta}{\gamma - 1} + \frac{\boldsymbol{V}^2}{2}.$$

So it remains to develop the jump on E^h in terms of the lagrangian ideal MHD conservative variables.

By the notations, we also have $E^h = E^* - \vartheta \boldsymbol{B}^2/2$. So we can write

$$\Delta E^h = \Delta E^* - \tilde{\vartheta} \Delta \left(\frac{\boldsymbol{B}^2}{2} \right) - \widetilde{\left(\frac{\boldsymbol{B}^2}{2} \right)} \Delta \vartheta$$
$$= \frac{(\boldsymbol{B}_l \cdot \boldsymbol{B}_r)}{2} \Delta \vartheta + \Delta E^* - \tilde{\boldsymbol{B}} \cdot \boldsymbol{\Delta(\vartheta B)}.$$

Finally, the jump on the full pressure is given by

$$\Delta p^* = \left(-\frac{\tilde{p}}{\tilde{\vartheta}} - \frac{(\tilde{\boldsymbol{B}})^2}{\tilde{\vartheta}} + \frac{\gamma - 1}{2} \frac{(\boldsymbol{B}_l \cdot \boldsymbol{B}_r)}{\tilde{\vartheta}} \right) \Delta \vartheta - \frac{\gamma - 1}{\tilde{\vartheta}} \tilde{\boldsymbol{V}} \cdot \Delta \boldsymbol{V} + \frac{\gamma - 1}{\tilde{\vartheta}} \Delta E^* + (2 - \gamma) \frac{\tilde{\boldsymbol{B}}}{\tilde{\vartheta}} \cdot \Delta(\vartheta \boldsymbol{B}). \tag{5.10}$$

In order to simplify this expression, we introduce two new variables ε^* and $\boldsymbol{B}^\perp$, defined by the relations

$$\varepsilon^* = \varepsilon + \vartheta \frac{\boldsymbol{B}^2}{2}, \quad \boldsymbol{B}^\perp = {}^t(0, B_y, B_z).$$

With the above notations and the assumption giving B_x as a real constant, the relation (5.10) becomes

$$\Delta p^* = \left(-\frac{\gamma - 1}{\tilde{\vartheta}} \widetilde{\left(\frac{\varepsilon^*}{\vartheta} \right)} + \frac{\gamma - 2}{\tilde{\vartheta}} (\tilde{\boldsymbol{B}}^\perp)^2 \right) \Delta \vartheta - \frac{\gamma - 1}{\tilde{\vartheta}} \tilde{\boldsymbol{V}} \cdot \Delta \boldsymbol{V} + \frac{\gamma - 1}{\tilde{\vartheta}} \Delta E^* + (2 - \gamma) \frac{\tilde{\boldsymbol{B}}^\perp}{\tilde{\vartheta}} \cdot \Delta(\vartheta \boldsymbol{B}). \tag{5.11}$$

The jump on the other components of the flux are developed by using the relations (5.7) to (5.11). Finally, we obtain the following matrix [22]:

$$\overline{A}_L = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ \xi_{21} & -(\gamma-1)\frac{\tilde{u}}{\tilde{\vartheta}} & -(\gamma-1)\frac{\tilde{v}}{\tilde{\vartheta}} & -(\gamma-1)\frac{\tilde{w}}{\tilde{\vartheta}} & (2-\gamma)\frac{\tilde{B}_y}{\tilde{\vartheta}} & (2-\gamma)\frac{\tilde{B}_z}{\tilde{\vartheta}} & \frac{(\gamma-1)}{\tilde{\vartheta}} \\ B_x\frac{\tilde{B}_y}{\tilde{\vartheta}} & 0 & 0 & 0 & -\frac{B_x}{\tilde{\vartheta}} & 0 & 0 \\ B_x\frac{\tilde{B}_z}{\tilde{\vartheta}} & 0 & 0 & 0 & 0 & -\frac{B_x}{\tilde{\vartheta}} & 0 \\ 0 & 0 & -B_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -B_x & 0 & 0 & 0 \\ \xi_{71} & \xi_{72} & \xi_{73} & \xi_{74} & \xi_{75} & \xi_{76} & \xi_{77} \end{bmatrix}, \tag{5.12}$$

where

$$\xi_{21} = \frac{(\gamma-2)}{\tilde{\vartheta}}(\tilde{\boldsymbol{B}}^{\perp})^2 - \frac{(\gamma-1)}{\tilde{\vartheta}}\left(\widetilde{\frac{\varepsilon^*}{\vartheta}}\right)$$

and

$$\xi_{71} = \frac{(\gamma-2)}{\tilde{\vartheta}}\tilde{u}\,(\tilde{\boldsymbol{B}}^{\perp})^2 - \frac{(\gamma-1)}{\tilde{\vartheta}}\tilde{u}\left(\widetilde{\frac{\varepsilon^*}{\vartheta}}\right) + \frac{B_x}{\tilde{\vartheta}}\left(\tilde{\boldsymbol{B}}^{\perp}\cdot\tilde{\boldsymbol{V}}\right)$$

$$\xi_{72} = \widetilde{p^*} - B_x^2 - \frac{(\gamma-1)}{\tilde{\vartheta}}\tilde{u}^2$$

$$\xi_{73} = -\frac{(\gamma-1)}{\tilde{\vartheta}}\tilde{u}\tilde{v} - B_x\tilde{B}_y, \quad \xi_{74} = -\frac{(\gamma-1)}{\tilde{\vartheta}}\tilde{u}\tilde{w} - B_x\tilde{B}_z$$

$$\xi_{75} = \frac{(2-\gamma)}{\tilde{\vartheta}}\tilde{u}\tilde{B}_y - \frac{B_x}{\tilde{\vartheta}}\tilde{v}, \quad \xi_{76} = \frac{(2-\gamma)}{\tilde{\vartheta}}\tilde{u}\tilde{B}_z - \frac{B_x}{\tilde{\vartheta}}\tilde{w},$$

$$\xi_{77} = \frac{(\gamma-1)}{\tilde{\vartheta}}\tilde{u}.$$

By construction this matrix satisfies the jump relations (5.6); it is consistent with the jacobian of the flux and it has a complete set of eigenvectors with the real eigenvalues

$$-\frac{\tilde{c}_f}{\tilde{\vartheta}}, -\frac{\tilde{c}_a}{\tilde{\vartheta}}, -\frac{\tilde{c}_s}{\tilde{\vartheta}}, 0, \frac{\tilde{c}_s}{\tilde{\vartheta}}, \frac{\tilde{c}_a}{\tilde{\vartheta}}, \frac{\tilde{c}_f}{\tilde{\vartheta}}$$

with

$$\tilde{c}_f^2 = \frac{1}{2}\left((\widetilde{a^*})^2 + \sqrt{(\tilde{a}^*)^4 - 4\tilde{a}^2\tilde{b}_x^2}\right),$$

$$\tilde{c}_s^2 = \frac{1}{2}\left((\widetilde{a^*})^2 - \sqrt{(\tilde{a}^*)^4 - 4\tilde{a}^2\tilde{b}_x^2}\right)$$

$$\tilde{c}_a^2 = \tilde{b}_x^2$$

$$\tilde{\boldsymbol{b}} = \tilde{\boldsymbol{B}}\sqrt{\tilde{\vartheta}}, \quad \tilde{\boldsymbol{b}} = {}^t(\tilde{b}_x, \tilde{b}_y, \tilde{b}_z)$$

$$\widetilde{a^{*2}} = \tilde{a}^2 + \tilde{\boldsymbol{b}}^2, \quad \tilde{a}^2 = \gamma P\tilde{\vartheta}$$

$$P = \tilde{p} + \frac{\gamma-1}{\gamma}\left((\widetilde{\boldsymbol{B}^2}) - (\tilde{\boldsymbol{B}})^2\right).$$

So $\overline{\boldsymbol{A}}_L$ is a Roe matrix for the lagrangian ideal MHD.

When $\gamma = 2$, it is interesting to note that this matrix is the jacobian evaluated at the averaged state:

$$\left(\tilde{\vartheta}, \tilde{\boldsymbol{V}}, p + \frac{\widetilde{\boldsymbol{B}^2}}{2}, \tilde{\boldsymbol{B}}\right). \tag{5.13}$$

5.3. *Numerical Results*

With the Roe matrix (5.12), we construct a first-order Roe-type scheme to solve the ideal MHD equations using lagrangian coordinates (5.2). We choose the same initial condition as the first numerical example (4.22) presented for the eulerian form.

On Fig. 3 and Fig. 4 are shown all the variables computed by the Roe type scheme and the Lax–Friedrichs scheme with 800 cells and a CFL number equal to 0.9. The numerical solution is presented at 100 s. The Roe scheme still gives better results than the Lax scheme.

It is interesting to note that the solution has the same structure for the eulerian and lagrangian system. Indeed, we find again two fast rarefaction waves, a slow compound wave and a slow shock for each quantity, with a contact discontinuity in addition for the density.

6. CONSTRUCTION OF ROE MATRICES FOR GENERAL SYSTEMS OF CONSERVATION LAWS

In this section, we are interested in the numerical resolution of general hyperbolic systems of conservation laws by Roe's schemes [23]. We try to construct Roe matrices for systems using eulerian or lagrangian coordinates. In the following, we describe how to compute systematically Roe matrices for one of these two forms in terms of the Roe matrix for the other form.

6.1. *Introduction*

Let Ω be a domain in IR^3. Its boundary denoted $\partial\Omega$ is followed in its movement which is characterized by the material velocity $\boldsymbol{V}$.

An integral conservation law for a quantity $\boldsymbol{U}$ associated to a flux $\boldsymbol{G}_0$ will be written

$$\frac{d}{dt}\int_{\Omega}\boldsymbol{U}\,d\Omega + \int_{\partial\Omega}\boldsymbol{G}_0\cdot\nu\,d\sigma = \boldsymbol{0},$$

where ν is the outward unit normal vector to Ω.

This integral form leads to the differential form using eulerian coordinates

$$\frac{\partial\boldsymbol{U}}{\partial t} + div(\boldsymbol{U}\otimes\boldsymbol{V} + G_0) = \boldsymbol{0},$$

where the velocity satisfies the relation

$$\frac{d\boldsymbol{M}}{dt} = \boldsymbol{V} = (u, v, w) \quad \text{for any point } \boldsymbol{M} \text{ in } IR^3.$$

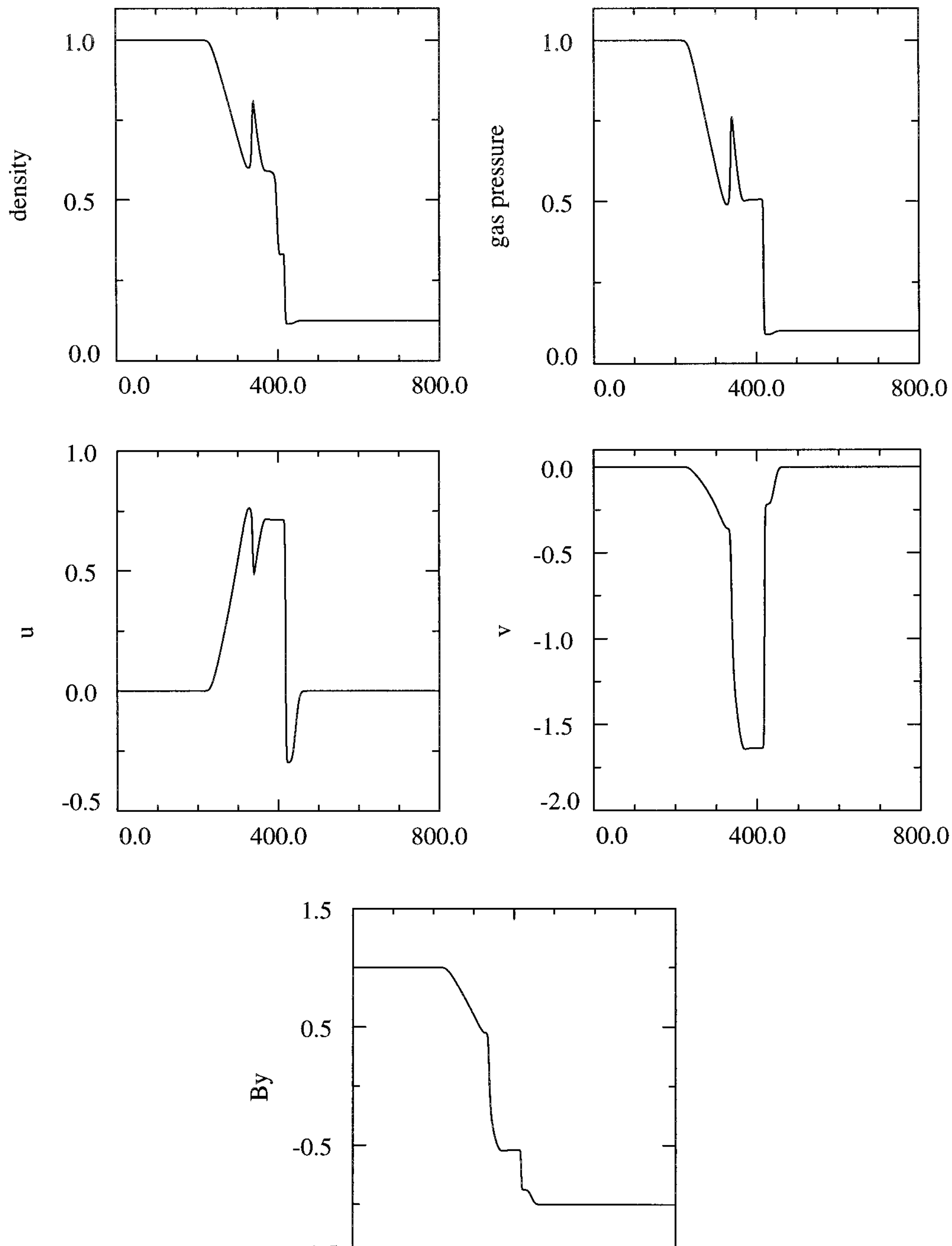

FIG. 3. Roe-type scheme for lagrangian coordinates.

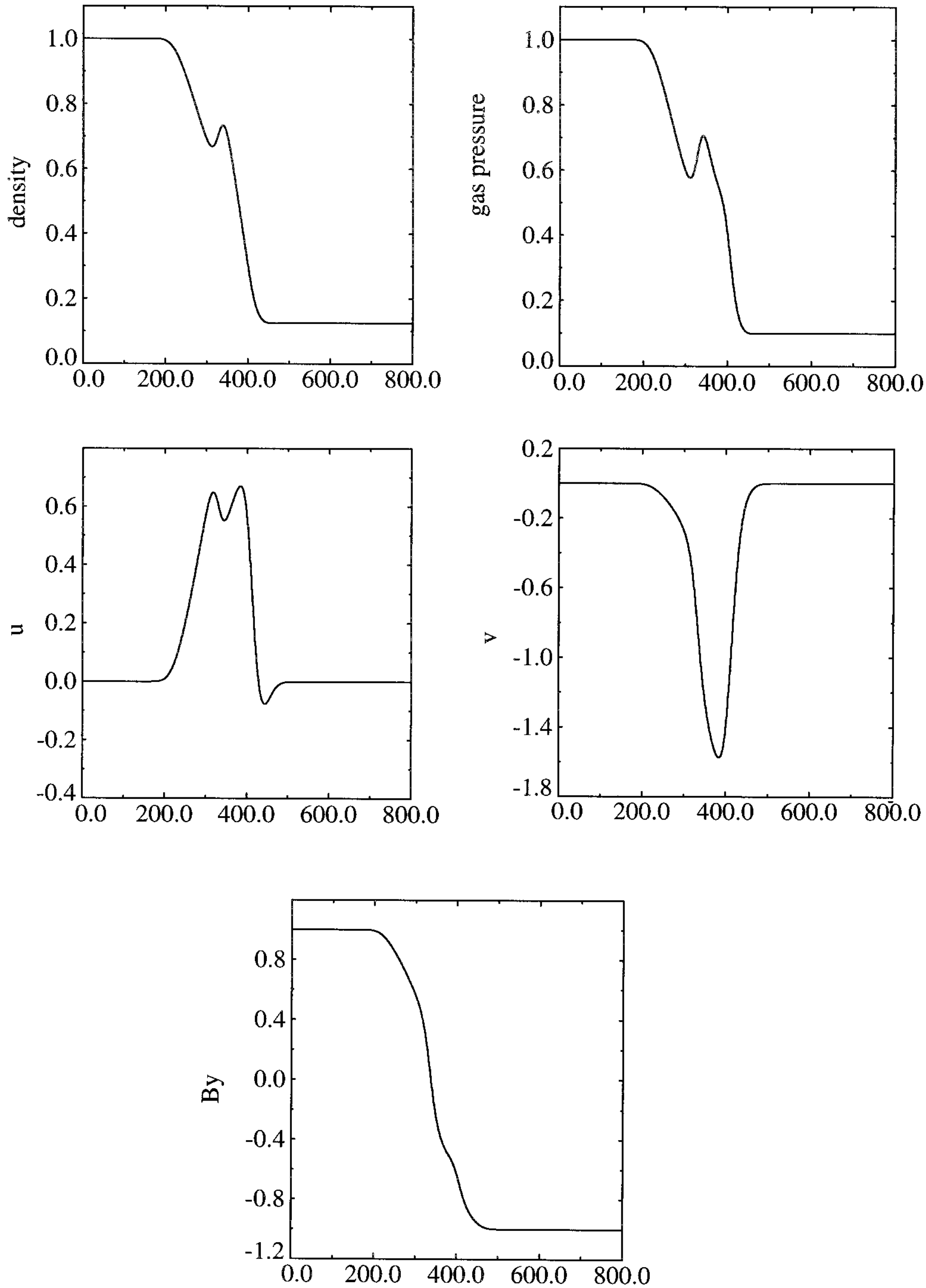

FIG. 4. Lax scheme for lagrangian coordinates.

In one space dimension, this relation becomes

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial x} = \boldsymbol{0} \tag{6.1}$$

with the following definition of the flux:

$$F = u\boldsymbol{U} + \boldsymbol{G}_0. \tag{6.2}$$

Now we suppose that the first component of the vector $\boldsymbol{U}$ is given by the density ρ, which satisfies

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\rho u = 0.$$

We remark that the first component of $\boldsymbol{G}_0$ is identically zero.

In order to obtain the lagrangian form, we use the same variables τ, m as defined in Section 5 by the relation (5.1) and we introduce the following quantities:

$$\begin{gathered} \boldsymbol{n} = {}^t(1, 0, 0, \ldots\ldots, 0), \quad \boldsymbol{U} = \rho\boldsymbol{n} + \boldsymbol{U}_0, \quad \boldsymbol{W}_0 = \frac{\boldsymbol{U}_0}{\rho}, \\ \boldsymbol{W} = \frac{1}{\rho}\boldsymbol{n} + \boldsymbol{W}_0, \quad \boldsymbol{G} = -u\boldsymbol{n} + \boldsymbol{G}_0. \end{gathered} \tag{6.3}$$

So we get the following conservation law for lagrangian coordinates:

$$\frac{\partial \boldsymbol{W}}{\partial \tau} + \frac{\partial \boldsymbol{G}}{\partial m} = \boldsymbol{0}. \tag{6.4}$$

6.2. *Relation between the Jacobians of the Conservative Fluxes*

It is easy to connect the two jacobians of the fluxes for the eulerian and lagrangian forms. Indeed, let us define the matrices

$$\boldsymbol{U}_{\boldsymbol{W}} = \frac{\partial \boldsymbol{U}}{\partial \boldsymbol{W}}, \quad \boldsymbol{W}_U = \frac{\partial \boldsymbol{W}}{\partial \boldsymbol{U}} = \boldsymbol{U}_W^{-1}, \tag{6.5}$$

and the jacobians of the eulerian and lagrangian fluxes,

$$\boldsymbol{A}_E = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}}, \quad \boldsymbol{A}_L = \frac{\partial \boldsymbol{G}}{\partial \boldsymbol{W}}. \tag{6.6}$$

So we have a first result:

PROPOSITION 1. *The jacobians $\boldsymbol{A}_E$ and $\boldsymbol{A}_L$ are connected by the simple relation*

$$\boldsymbol{A}_L = -\rho u\boldsymbol{I} + \rho\boldsymbol{W}_{\boldsymbol{U}}\boldsymbol{A}_E\boldsymbol{U}_{\boldsymbol{W}}, \tag{6.7}$$

where $\boldsymbol{I}$ denotes the identity matrix.

Proof. From the definition of the eulerian flux (6.2), we get the differential relation:

$$\boldsymbol{A}_E\, d\boldsymbol{U} = d\boldsymbol{F} = u\, d\boldsymbol{U} + \boldsymbol{U}\, du + d\boldsymbol{G}_0.$$

Then, we multiply the above equation on the left by the matrix $\boldsymbol{W}_{\boldsymbol{U}}$, and we obtain

$$\boldsymbol{W}_{\boldsymbol{U}}\boldsymbol{A}_E\boldsymbol{U}_{\boldsymbol{W}}\, d\boldsymbol{W} = u\, d\boldsymbol{W} + \boldsymbol{W}_{\boldsymbol{U}}\boldsymbol{U}\, du + \boldsymbol{W}_{\boldsymbol{U}}\, d\boldsymbol{G}_0. \tag{6.8}$$

By the definitions (6.3) and (6.5), it is easy to show the relation

$$d\boldsymbol{W} = \boldsymbol{W}_U\, d\boldsymbol{U} = -\frac{\boldsymbol{W}}{\rho}\, d\rho + \frac{1}{\rho}\, d\boldsymbol{U}_0$$

which yields the identities

$$\boldsymbol{W}_{\boldsymbol{U}}\boldsymbol{U} = -\frac{1}{\rho}\boldsymbol{n}$$

$$\boldsymbol{W}_{\boldsymbol{U}} d\boldsymbol{G}_0 = \frac{1}{\rho}\, d\boldsymbol{G}_0.$$

Finally, by inserting these two relations in (6.8), we get

$$\begin{aligned} \boldsymbol{W}_{\boldsymbol{U}}\boldsymbol{A}_E\boldsymbol{U}_{\boldsymbol{W}}\, d\boldsymbol{W} &= u\, d\boldsymbol{W} - \frac{1}{\rho}\boldsymbol{n}\, du + \frac{1}{\rho}\, d\,\boldsymbol{G}_0 \\ &= u\, d\boldsymbol{W} + \frac{1}{\rho}\, d\boldsymbol{G} = \left(u\boldsymbol{I} + \frac{1}{\rho}\boldsymbol{A}_L\right) d\boldsymbol{W} \end{aligned}$$

which implies (6.7). ▮

6.3. *Relation between the Roe Matrices of the Eulerian and Lagrangian Forms*

We introduce a new kind of average associated to a parameter a, which is defined by the relation

$$\xi_a = a\xi_l + (1 - a)\xi_r. \tag{6.9}$$

If we define b such that: $a + b = 1$, it is interesting to note that we have the general identity

$$\Delta(xy) = x_a\, \Delta y + y_b\, \Delta x. \tag{6.10}$$

Let us note that Eq. (4.6) can be obtained from the relation (6.9) with the choice of a,

$$a = \frac{\sqrt{\rho_l}}{\sqrt{\rho_l} + \sqrt{\rho_r}}, \tag{6.11}$$

and the arithmetic average is given for $a = \frac{1}{2}$.

In the same way as in the continuous case, we establish a relation which looks like (6.7) and connects the Roe matrices of the eulerian and lagrangian forms. For this purpose, we introduce the notion of discrete transformation matrix from eulerian coordinates to lagrangian coordinates with the following definition.

DEFINITION. A matrix $\overline{\boldsymbol{W}}_U$ is called an *Euler–Lagrange discrete transformation matrix* if it satisfies the following properties:

(i) it is invertible

(ii) it satisfies: $\Delta \boldsymbol{W} = \overline{\boldsymbol{W}}_U \, \Delta \boldsymbol{U}$.

The inverse matrix, denoted $\overline{\boldsymbol{U}}_{\boldsymbol{W}}$, is called a *Lagrange–Euler discrete transformation matrix.*

By the identity (6.10), we have

$$\Delta \boldsymbol{W} = \vartheta_a \, \Delta(\rho \boldsymbol{W}) + \Delta \vartheta (\rho \boldsymbol{W})_b$$

which can be written, with the definitions (6.3),

$$\Delta \boldsymbol{W} = \vartheta_a \, \Delta \boldsymbol{U}_0 - \frac{1}{\rho_l \rho_r} (\rho \boldsymbol{W})_b \, \Delta \rho.$$

This relation involves an Euler–Lagrange discrete transformation matrix, denoted $\overline{\boldsymbol{W}}{}^a_U$, where a is a parameter. It satisfies the relation

$$\overline{\boldsymbol{W}}{}^a_U \boldsymbol{X} = -\frac{x}{\rho_l \rho_r} (\rho \boldsymbol{W})_b + \vartheta_a \boldsymbol{X}_0 \tag{6.12}$$

for any vector $\boldsymbol{X}$ defined by $\boldsymbol{X} = x\boldsymbol{n} + \boldsymbol{X}_0$, with $\boldsymbol{X}_0$ such that its first component is equal to zero.

Now we give a second result:

PROPOSITION 2. *Let us suppose that a Roe matrix $\overline{\boldsymbol{A}}_E$ is known for the eulerian form. Then an infinite number of Roe matrices for the lagrangian form exists. All these matrices, denoted $\overline{\boldsymbol{A}}{}^a_L$, are parametrized by a real a. They are given by the following identity:*

$$\overline{\boldsymbol{A}}{}^a_L = \vartheta_a^{-1} (-u_a \boldsymbol{I} + \overline{\boldsymbol{W}}{}^a_U \overline{\boldsymbol{A}}_E \overline{\boldsymbol{U}}{}^a_{\boldsymbol{W}}). \tag{6.13}$$

Conversely, if one knows a Roe matrix $\overline{\boldsymbol{A}}_L$ for the lagrangian form, we can construct an infinite number of Roe matrices for the eulerian form. They are given by the relation:

$$\overline{\boldsymbol{A}}{}^a_E = u_a \boldsymbol{I} + \vartheta_a \overline{\boldsymbol{U}}{}^a_{\boldsymbol{W}} \overline{\boldsymbol{A}}_L \overline{\boldsymbol{W}}{}^a_U. \tag{6.14}$$

Proof. By construction, we have

$$\Delta \boldsymbol{F} = \overline{\boldsymbol{A}}_E \, \Delta \boldsymbol{U} = \overline{\boldsymbol{A}}_E \, \overline{\boldsymbol{U}}{}^a_{\boldsymbol{W}} \Delta \boldsymbol{W}$$

which yields

$$\overline{\boldsymbol{A}}_E \overline{\boldsymbol{U}}{}^a_{\boldsymbol{W}} \Delta \boldsymbol{W} = u_a \, \Delta \boldsymbol{U} + \boldsymbol{U}_b \, \Delta u + \Delta \boldsymbol{G}_0$$

by the definition (6.2) of the flux.

We multiply this identity by $\overline{\boldsymbol{W}}{}^a_U$ on the left and we get

$$\overline{\boldsymbol{W}}{}^a_{\boldsymbol{U}} \overline{\boldsymbol{A}}_E \overline{\boldsymbol{U}}{}^a_{\boldsymbol{W}} \, \Delta \boldsymbol{W} = u_a \, \Delta \boldsymbol{W} + \Delta u \overline{\boldsymbol{W}}{}^a_{\boldsymbol{U}} \boldsymbol{U}_b + \overline{\boldsymbol{W}}{}^a_U \, \Delta \boldsymbol{G}_0.$$

Now we apply the relation (6.12) to the vectors $\boldsymbol{U}_b$ and $\Delta \boldsymbol{G}_0$, instead of $\boldsymbol{X}$, and we find

$$\overline{\boldsymbol{W}}{}^a_U \boldsymbol{U}_b = -\vartheta_a \boldsymbol{n},$$
$$\overline{\boldsymbol{W}}{}^a_U \, \Delta \boldsymbol{G}_0 = \vartheta_a \, \Delta \boldsymbol{G}_0.$$

Finally, it follows that

$$\overline{\boldsymbol{W}}{}^a_{\boldsymbol{U}} \overline{\boldsymbol{A}}_E \overline{\boldsymbol{U}}{}^a_{\boldsymbol{W}} \, \Delta \boldsymbol{W} = u_a \, \Delta \boldsymbol{W} + \vartheta_a (-\Delta u \boldsymbol{n} + \Delta \boldsymbol{G}_0)$$

which implies the desired identity. ▮

Remark. Let us note that by recurrence it is easy to extend this result [23] and to show the following:

If there exists an n-parameter family of Roe matrices for one of the two forms, then there exists an n-parameter family for the other form.

We conclude this section by giving some properties on the eigenvalues and the eigenvectors for the eulerian and lagrangian forms. Indeed, it is interesting to remark that simple relations connect these quantities for one of the two forms to the same quantities for the other form.

Let us note λ_L, $\boldsymbol{R}_L$, $\boldsymbol{L}_L$, and η_L the eigenvalues, the right and left eigenvectors, and the characteristic variables for the Roe matrix of the lagrangian form. λ_E, $\boldsymbol{R}_E$, $\boldsymbol{L}_E$, and η_E denote the same quantities for the eulerian form.

By using the relations between the Roe matrices of the two forms (cf. Proposition 2), we can easily establish the following identities:

$$\begin{aligned} \lambda_L &= \vartheta_a^{-1} (\lambda_E - u_a) \\ \boldsymbol{R}_L &= \overline{\boldsymbol{W}}{}^a_U \boldsymbol{R}_E \\ \boldsymbol{L}_L &= \boldsymbol{L}_E \overline{\boldsymbol{U}}{}^a_{\boldsymbol{W}}. \end{aligned} \tag{6.15}$$

Moreover, by the definitions of the characteristic variables and the Lagrange–Euler discrete transformation matrix, we show that the characteristic variables are the same for the two forms:

$$\eta_L = \boldsymbol{L}_L \, \Delta \boldsymbol{W} = \boldsymbol{L}_E \, \Delta \boldsymbol{U} = \eta_E. \tag{6.16}$$

7. APPLICATIONS

In this section, we describe the application of the general results obtained in the last section for the gas dynamics equations and the ideal MHD model.

7.1. *Application to Gas Dynamics*

First, let us recall the model. When eulerian coordinates are used, the system is given by the set of equations,

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\rho u &= 0 \\ \frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + p) &= 0 \\ \frac{\partial}{\partial t}(\rho E) + \frac{\partial}{\partial x}(\rho u H) &= 0, \end{aligned} \tag{7.1}$$

where

$$\rho E = \frac{1}{2}\rho u^2 + \frac{p}{\gamma - 1}, \quad \rho H = \rho E + p,$$

and the lagrangian gas dynamics model satisfies

$$\begin{aligned} \frac{\partial \vartheta}{\partial \tau} - \frac{\partial u}{\partial m} &= 0 \\ \frac{\partial u}{\partial \tau} + \frac{\partial p}{\partial m} &= 0 \\ \frac{\partial E}{\partial \tau} + \frac{\partial}{\partial m}(up) &= 0. \end{aligned} \tag{7.2}$$

For each form of the model, a Roe matrix has been constructed [12, 15]. Each one involves two different kinds of averages: arithmetic averages for the lagrangian one [12] and classical Roe averages for the eulerian one. We can apply the results of the sixth section to each of these Roe matrices. So we get an infinity of Roe matrices for the other form which are parametrized by a real a.

Especially, to obtain an eulerian Roe matrix, we choose the parameter a and the discrete transformation matrices such that:

$$\xi_a = \tilde{\xi}. \tag{7.3}$$

In this case, we have

$$\overline{\boldsymbol{U}}_{\boldsymbol{W}} = \frac{1}{\tilde{\vartheta}}\begin{bmatrix} -\tilde{\rho} & 0 & 0 \\ -(\widetilde{\rho u}) & 1 & 0 \\ -(\widetilde{\rho E}) & 0 & 1 \end{bmatrix}, \overline{\boldsymbol{W}}_{\boldsymbol{U}} = \frac{1}{\rho_l \rho_r}\begin{bmatrix} -1 & 0 & 0 \\ -(\widetilde{\rho u}) & \tilde{\rho} & 0 \\ -(\widetilde{\rho E}) & 0 & \tilde{\rho} \end{bmatrix}. \tag{7.4}$$

Then we apply the relation (6.14) to the lagrangian Roe matrix of Munz, which yields a new Roe matrix for the eulerian gas dynamics:

$$\begin{bmatrix} \tilde{u} - \tilde{\vartheta}(\widetilde{\rho u}) & \tilde{\rho}\tilde{\vartheta} & 0 \\ -\frac{(\widetilde{\rho u})^2}{\rho_l \rho_r} + \frac{\gamma - 1}{2} u_l u_r & (2-\gamma)\tilde{u} + \tilde{\vartheta}(\widetilde{\rho u}) & \gamma - 1 \\ -\frac{(\widetilde{\rho u})}{\rho_l \rho_r}(\widetilde{\rho H}) + \frac{\gamma - 1}{2} u_l u_r \tilde{u} & \tilde{\vartheta}(\widetilde{\rho H}) - (\gamma - 1)\tilde{u}^2 & \gamma \tilde{u} \end{bmatrix}. \tag{7.5}$$

Its eigenvalues are given by

$$\tilde{u} - \tilde{c}, \tilde{u}, \tilde{u} + \tilde{c} \quad \text{with } \tilde{c} = \sqrt{\gamma \tilde{p} \tilde{\vartheta}}.$$

The identities (6.15) lead to the expressions for the right eigenvectors,

$$\boldsymbol{R}_{\tilde{u}-\tilde{c}} = \frac{1}{\tilde{\rho}\tilde{a}^2}\begin{bmatrix} \tilde{\rho} \\ (\widetilde{\rho u}) - \tilde{C} \\ (\widetilde{\rho H}) - \tilde{u}\tilde{C} \end{bmatrix}, \quad \boldsymbol{R}_{\tilde{u}} = \frac{1}{\tilde{\rho}\tilde{a}^2}\begin{bmatrix} \tilde{\rho} \\ (\widetilde{\rho u}) \\ \left(\widetilde{\frac{\rho u^2}{2}}\right) \end{bmatrix},$$

$$\boldsymbol{R}_{\tilde{u}+\tilde{c}} = \frac{1}{\tilde{\rho}\tilde{a}^2}\begin{bmatrix} \tilde{\rho} \\ (\widetilde{\rho u}) + \tilde{C} \\ (\widetilde{\rho H}) + \tilde{u}\tilde{C} \end{bmatrix},$$

and for the characteristic variables,

$$\alpha_{\tilde{u}\pm\tilde{c}} = \tfrac{1}{2}(\Delta p \pm \tilde{C}\,\Delta u), \quad \alpha_{\tilde{u}} = \tilde{a}^2\,\Delta\rho - \Delta\rho,$$

where

$$\tilde{C} = \frac{\tilde{c}}{\tilde{\vartheta}} = \sqrt{\gamma\frac{\tilde{p}}{\tilde{\vartheta}}}, \quad \tilde{a}^2 = \gamma\frac{\tilde{p}}{\tilde{\rho}}.$$

Let us note that this matrix involves only one kind of averages, arithmetic ones, unlike the classical eulerian Roe matrix which contains Roe averages. Moreover, if we look at the first line of the matrix (7.5), we see that it gives the surprising decomposition,

$$\Delta(\rho u) = (\tilde{u} - \tilde{\vartheta}(\widetilde{\rho u}))\,\Delta\rho + \tilde{\rho}\tilde{\vartheta}\,\Delta(\rho u),$$

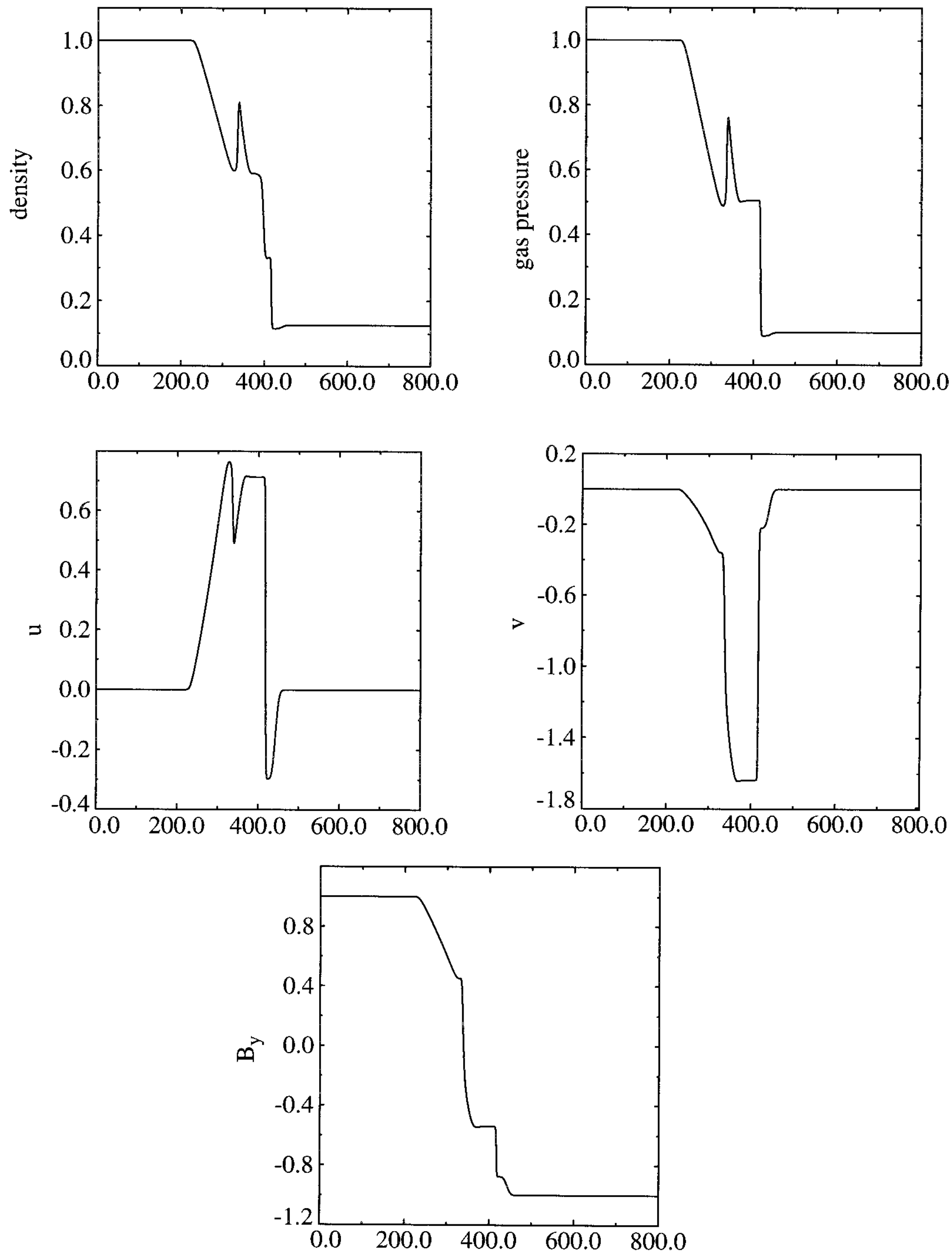

FIG. 5. Lagrangian Roe matrix with Roe averages.

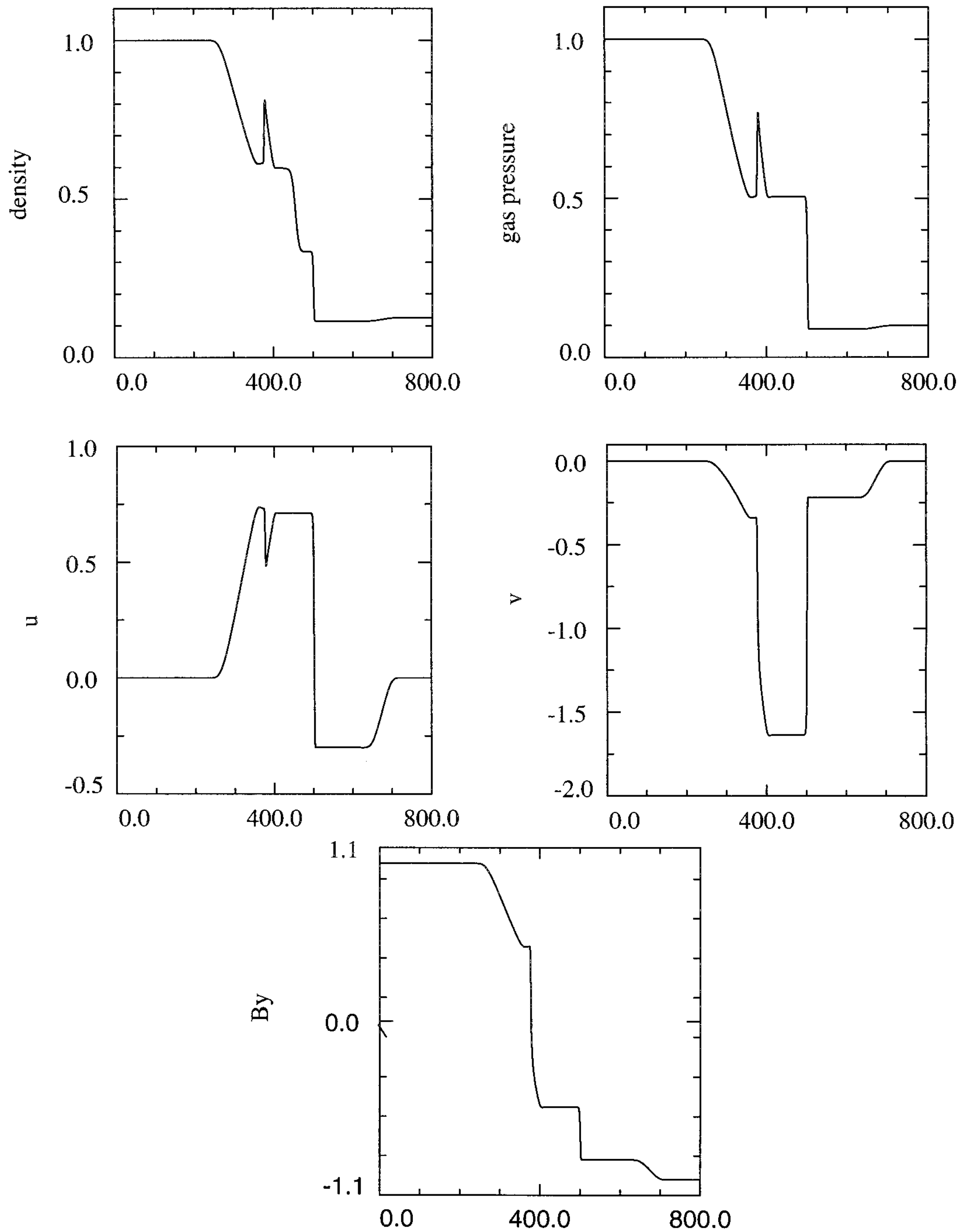

FIG. 6. Eulerian Roe matrix with arithmetic averages.

instead of

$$\Delta(\rho u) = \Delta(\rho u)$$

which is more natural.

It is interesting to remark that the sound speeds involved by this matrix only depend on the thermodynamic variables.

To conclude this section, numerical experiments with this Roe matrix show no difference with the results obtained with the classical Roe matrix.

7.2. *Application to Ideal MHD*

In sections 4 and 5, we have computed two Roe matrices for ideal MHD, one for the eulerian form and the other one for the lagrangian form. We can apply the results of the last section to construct new Roe matrices for these two forms.

Especially, from the eulerian Roe matrix, we choose the real parameter a like in (6.11),

$$a = \frac{\sqrt{\rho_l}}{\sqrt{\rho_l} + \sqrt{\rho_r}},$$

and we get a lagrangian Roe matrix whose eigenvalues are

$$-\underline{\rho}\bar{c}_s, -\underline{\rho}\bar{c}_a, -\underline{\rho}\bar{c}_f, 0, \underline{\rho}\bar{c}_s, \underline{\rho}\bar{c}_a, \underline{\rho}\bar{c}_f,$$

where $\bar{c}_s$, $\bar{c}_a$, $\bar{c}_f$ are defined by (4.17).

Unlike the Roe matrix computed in Section 5 with arithmetic averages, this one does not define a Roe average even for the special case $\gamma = 2$.

We use this matrix to construct a Roe-type scheme to solve lagrangian ideal MHD. Figure 5 presents the results obtained on the same Riemann problem as for Section 5. A comparison between these two lagrangian Roe type schemes shows that they give exactly the same results.

Conversely, we can construct a new eulerian Roe matrix for ideal MHD from the lagrangian Roe matrix based on the extension of Munz's results and given in Section 5. The resulting matrix involves arithmetic averages like the eigenvalues which are given by

$$\tilde{u} - \tilde{c}_f, \tilde{u} - \tilde{c}_a, \tilde{u} - \tilde{c}_s, \tilde{u}, \tilde{u} + \tilde{c}_s, \tilde{u} + \tilde{c}_a, \tilde{u} + \tilde{c}_f.$$

On Fig. 6, we present the numerical experiment given by the Riemann problem (4.22) of the fourth section with the Roe-type scheme constructed with this Roe matrix.

We can note that no difference exists between the results obtained with the two matrices. Moreover, although the matrix (4.16) and the new matrix deduced from Munz's matrix involve different averages, the costs of the calculation only differ in a few percentages in favour of the new one.

Remark. The numerical examples presented on Fig. 5 and Fig. 6 are computed with an 800 cell mesh. This mesh is quite fine. In fact, numerical simulations made with 200 cells do not show any difference between all the schemes.

8. CONCLUSION

In this paper, several results about Roe matrices for a system of hyperbolic conservation laws are presented. The first result deals with the eulerian MHD system and constitutes a real improvement for the calculation of MHD flows. Indeed, in opposition to the result of Brio and Wu, who found that a Roe matrix exists for the special case $\gamma = 2$ and is a jacobian at an averaged state, our matrix is obtained without any hypothesis on γ. As demonstrated before, its construction is based on an original relation which expresses the magnetic pressure jump in terms of the density jump.

The second one is for the ideal MHD system in lagrangian coordinates. In fact, in the same fashion that Munz derived a Roe matrix for the lagrangian gas dynamics system, a Roe matrix for lagrangian MHD is obtained. It is based on arithmetic averages. As for eulerian coordinates, the case $\gamma = 2$ is particular in the sense that the matrix is a jacobian at an averaged state.

The few calculations presented above show that even for first order, the Roe scheme is a very robust and accurate scheme for MHD in eulerian or lagrangian coordinates.

Although it is not shown here, extension to second order can be made using classical arguments to improve accuracy [28]. Moreover, as for the gas dynamics system, the construction of a Roe matrix for MHD with a general equation of state or a multispecies model can be obtained; the ideas of Glaister [29], Liou, van Leer, and Shuen [30], Vinokur and Montagne [31], Liu and Vinokur [32], can be extended in a straightforward manner to MHD in eulerian or lagrangian coordinates [22].

In addition, a more interesting result has been obtained for a two-temperature model which includes the ideal MHD equations for ions and an advection equation on the electronic entropy [24, 25]. The details can be found in [24], where they are given for eulerian and lagrangian coordinates.

Then the final result shown in this paper is the establishment of a general relation to construct an infinity of Roe matrices for an eulerian or lagrangian system from a Roe matrix which would be known for one of these two forms. Here, some applications of this relation to compute two new kinds of Roe matrices for the gas dynamics and the ideal MHD are presented: the first type is based on the classical Roe average and the second one on arithmetic averages. But we believe that this relation can have larger applications. An example could be to use it in order to construct eulerian Roe matrices for models whose eulerian form is very complex. Indeed, as the lagrangian form is generally simpler than the eulerian form, the computation of a lagrangian Roe matrix would be easy and would give an eulerian Roe matrix by the transformation relations. In a future paper [26] such an application on an MHD-like system will be presented.

We conclude on a very important application of these results: our Roe matrix can be extended to compute multidimensional MHD flows. In [10] Powell describes how to modify the MHD system to hold some important properties and, in particular, galilean invariance. His technique leads to the construction of modified MHD system. In [25, 26] it is shown how the matrix (4.16) can be naturally extended to multidimensionnal situations in such a way that the dissipation matrix satisfies the galilean invariance. An interesting interpretation, which leads to a natural discretization of Powell source term, is also given.

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